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LETTER TO THE EDITOR

Structure of generating partitions for two-dimensional maps

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Abstract. We discuss the non-uniqueness of the generating partitions of two-dimensional maps, and we give new explanations for ambiguities that are to be expected in the construction of generating partitions if a parameter is varied. Both points are discussed with the help of the connecting lines of homoclinic tangencies in the entire phase space, i.e. also outside the attractor, that can be calculated very easily with a new algorithm introduced recently.

Direct methods to calculate the entropy of a chaotic system rely on the concept of generating partitions (GP) (for an overview, see [1]). According to the conjecture formulated in [2] and motivated by the concepts of the critical points in maps on the interval [3], the dividing lines of the partition should be constructed by connecting homoclinic tangencies, i.e. points where the stable and unstable manifold are tangential. Nearby points on the same stable and unstable manifold in the vicinity of a homoclinic tangency attract each other under both forward and backward iteration. They can only be separated in a symbolic representation if the partition line passes through the homoclinic tangency or one of its (pre)images. By this method a two-element partition of the Hénon map for different parameters was constructed, which was shown to be generating, i.e. the so-defined symbolic dynamics yielded the correct metric entropy and did not code any two different periodic orbits the same way. The method was proven to be applicable in many more general circumstances [4, 5]. There are, however, two important questions that remain to be answered even from a heuristic point of view.

While locally the argument presented in [2] separates any two points, it is not clear how the homoclinic tangencies should be consistently connected along the entire attractor. There is no argument *a priori* concerning which of the infinitely many homoclinic tangencies (HTs) should be connected by a line such that the attractor is partitioned in a reasonable way. If there exists such a line defining a GP, it is not unique by far (e.g. obviously the images and preimages of such a line also yield generating partitions). Heuristic arguments suggest that lines between the primary HTs (PHTs) characterized by the minimal curvature of stable and unstable manifold are the best candidates for a GP. In some situations it is not clear, however, which of the HTs are primary. We will show that certain other candidates of lines between non-primary HTs also yield GPs. One main purpose of this letter is to present an *a priori* distinction rule between generating and non-generating partitions.

The second open problem for the construction of a GP was discovered in [6, 7]: At certain parameters PHTs disappear, and a discontinuity in the connection of HTs occurs.

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The coding of trajectories cannot be performed continuously as the parameter is varied in that regime. We will give a new explanation of this phenomenon illustrating the results in [6] from a different point of view. As was recently shown [5], in Hamiltonian systems (where the attractor fills the entire phase space) connecting the HTs becomes a problem through the presence of avoided crossings between lines of HTs *on the attractor*.

Both points that we want to discuss in this letter can be approached by considering the structure of the HTs located in the entire phase space, i.e. also outside the attractor. These structures can give us valuable information about how the HTs need to be connected in order to represent a GP, and in certain bifurcation situations (as the one described in [6]) the extension into the neighbourhood of the attractor can give us insights into the ambiguities of defining GPs. Our analysis in this letter is exclusively devoted to binary partitions of the Hénon map

$$x_{n+1} = a - x_n^2 + by_n \qquad y_{n+1} = x_n \tag{1}$$

(with metric entropy smaller than $\ln 2$) but it can easily be extended to other systems as well.

We first want to briefly recall our method to determine the HTs outside the attractor. We use the method described in detail in [8] based on the local calculation of the most expanding direction in tangent space. The HTs are uniquely characterized by the fact that $\boldsymbol{e}_n^m(\boldsymbol{x})$, the most expanding direction in the tangent space over n iterations with the local Jacobians Df, is perpendicular to the tangent $E_u(\boldsymbol{x})$ of the attractor (unstable manifold) for large enough n. In fact, this statement holds precisely only in the case $n \to \infty$. However, in practice a finite number of iterations is sufficient as was shown in [8, 9]. The vector $\boldsymbol{e}_n^m(\boldsymbol{x})$ can be calculated easily as an eigenvector of $[Df^n(\boldsymbol{x})]^\dagger Df^n(\boldsymbol{x})$, where $[Df^n(\boldsymbol{x})]^\dagger$ denotes the transpose matrix of $Df^n(\boldsymbol{x})$. Figure 1 shows the structure of the HTs located in the entire phase space of the Hénon system for the standard parameters $(a, b) = (1.4, 0.3)$. Following the lines of HTs we clearly observe what is baptized 'avoided crossing' (AC) at different positions in phase space. A very similar AC structure of the connecting lines of HTs is observed in conservative maps where the ACs cause significant problems in the construction of a GP [5]. We also observed the ACs for the Ikeda system [10] with a more complicated structure.

The occurrence of an AC has a very simple geometrical reason. Considering the stable foliation locally as a set of parabolae meeting another set of parabolae of the unstable foliation, the points of tangency between those two sets lie on hyperbolae thus forming an AC. As we see in figure 1, no AC is located on the attractor and thus the definition of a GP should not cause any trouble. Note that the ACs are images of each other, so that iterating the regions of ACs backwards we see a more detailed version of the AC structure. For example, the ACs near the points labelled C_{-1} and D_{-1} in figure 2 are the preimages of the ACs near C_0 and D_0, which are no longer so well observable. In other words, the AC structure appears on all different length scales in the phase space.

In [6] how a PHT disappears at a certain critical parameter $a_c (= 1.356\,9288)$ ($b = 0.3 = \text{constant}$) was studied and, as a consequence, a discontinuity in the construction of the generating partition was invoked [6]. We state that this bifurcation is reflected in the fact that there occurs an AC of lines of HTs on the attractor itself. This means that there is an infinite number of ACs on the attractor. Under variation of a the AC 'moves over' the attractor. Indeed, figure 3 shows this very clearly. For $a > a_c$ (figure 3(a)) and $a < a_c$ (figure 3(c)) the AC is not located on the attractor, but on different sides of the attractor. For $a = a_c$ (figure 3(b)) the AC is located right on the attractor. Note that this structure also appears on all preimages and images. The problem for the construction of a GP in this situation

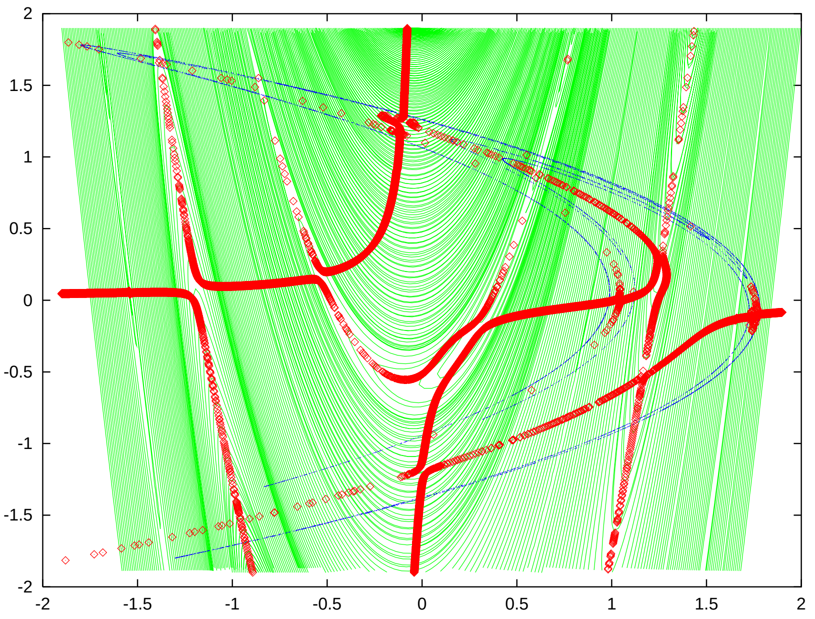


Figure 1. Homoclinic tangencies for the Hénon map in the entire phase space obtained by the method of finding the most contracting directions [8]. Additionally, the attractor and the stable foliation are shown. In [8] this figure can be seen in colour.

is how to connect the HT on the different branches of the attractor, a problem well-known in conservative systems, where the attractor fills the entire phase space. The ambiguities and difficulties of constructing a generating partition in the latter case are discussed in detail in [5, 11]. However, as we see even in dissipative systems, the occurrence of AC on the attractor can lead to the kind of ambiguities described in [6, 7]: a discontinuity in the construction of a generating partition. All trajectories passing through a critical region on the attractor undergo a discontinuous change in their symbolic dynamics, when the parameter is slightly changed. This problem has been treated pragmatically in a satisfying way [6], but *a priori* it does not seem possible to give a non-ambiguous construction of a generating partition.

However, even if there is no AC of lines of HTs located on the attractor, there is no *unique* construction of a GP. As we will show now, there are several different GPs possible, and we want to give an *a priori* criteria, when a given partition of the phase space is generating, and when it is not.

Figure 2 shows the Hénon attractor with the most relevant HTs labelled with letters that we will refer to in the following (although there are an infinite number of families of HTs, we label only the four families that are resolved by their PHTs, see figure 2). Evidently two partitions are equivalent, if the partition line passes through the same points on the attractor in the same order. Each partition shall be represented by that partition line that follows the HTs in the entire phase space and is denoted by the order of the HTs on the attractor that it passes through. Looking at figure 1 we observe that in order to define such a partition line we hit ACs at some points. For the standard partition [A_0, B_0, C_0, D_0] [2]

in figure 4 I this is clearly observable. The questions when constructing a GP by following such lines of HTs in the phase space is how to consider the ACs and where to follow the line continuously, where and how to jump over the AC? We obviously have three different choices at each AC of how to continue with the partition line.

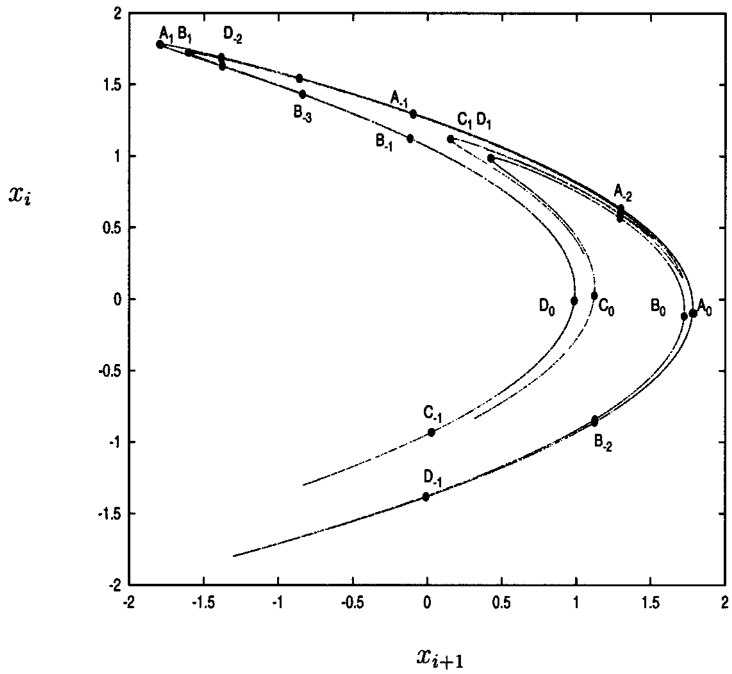


Figure 2. Homoclinic tangencies (labelled and referred to in the text) for the Hénon map. Note that we have indicated only four (groups of) HTs. Each of these groups consists of an infinite number of HTs (e.g. A_{-2} resolves several more HTs than A_0).

A given PHT represents an entire bi-infinite sequence of HTs, which is given by the images and preimages of the PHT. Passing through any combination of HTs in order to locally separate nearby points that are located on the same stable and unstable manifold [2, 5] may be globally inconsistent, for example by leading to a too small probability of one symbol or symbol sequence to occur. Figure 4 gives nine different partitions of the Hénon system with standard parameters $(a, b) = (1.4, 0.3)$ for which we have calculated the incremental entropies $h_n = H_{n+1} - H_n$. The value for the metric entropy obtained by calculating the positive Lyapunov exponent is $h_{\mathrm{metric}} = 0.419\,38$. Figure 5 shows h_n^i for the different partitions $i =$ I–IX (where n denotes the length of the symbol sequence). For a GP we expect

$$h_{29} \approx \lim_{n\to\infty} h_n = h_{\mathrm{metric}}. \tag{2}$$

The value obtained by calculating the Lyapunov exponent is indicated by the horizontal line.

Figure 4 II shows the partition [A_0, B_0, D_{-1}]. The numerical value of h_{29} is $h_{29} = 0.342\,676$, which is obviously much too small. The partition line does not pass through representatives of all families. Figure 4 III shows the partition [A_{-1}, C_1, D_1, B_0, A_0], which gives the value $h_{29} = 0.4079 < h_{\mathrm{metric}}$. A first necessary condition for a GP is therefore conjectured:

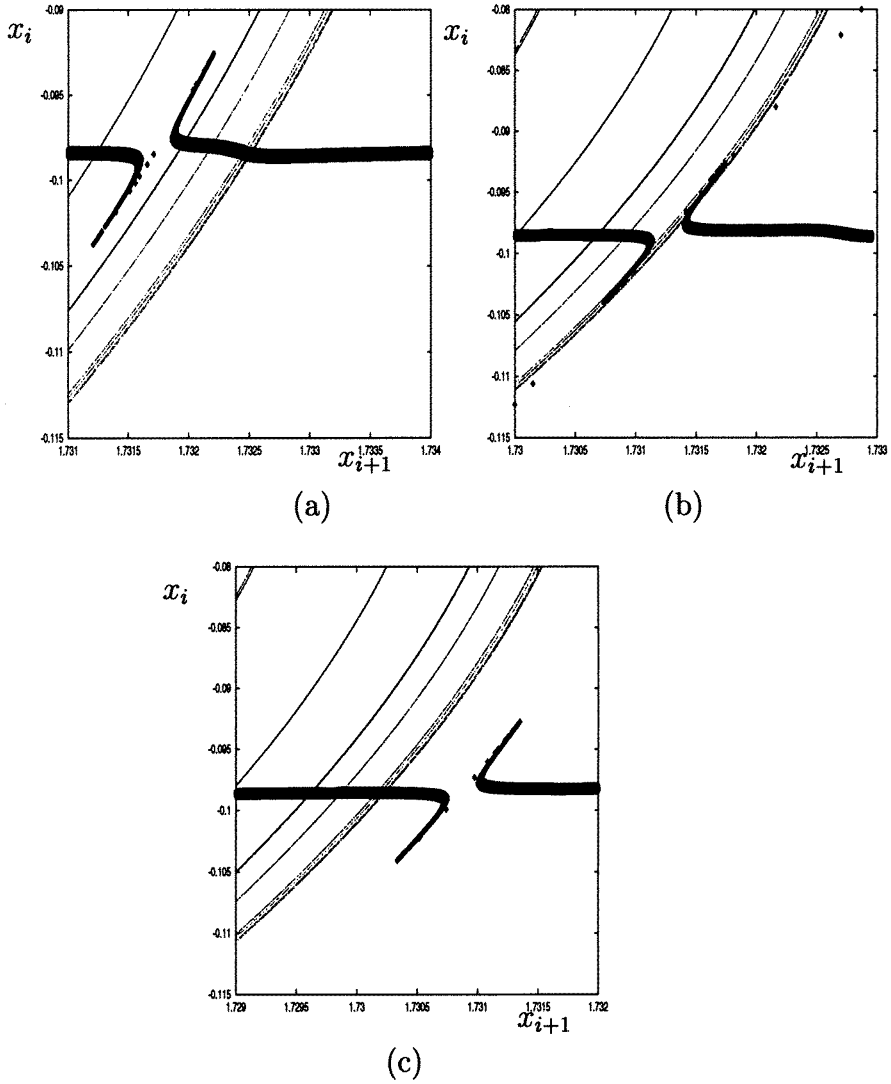


Figure 3. AC of lines of HT as the parameter a in (1) is varied in the region of $a_c = 1.356\,9288$. (a) $a > a_c$, (b) $a = a_c$ and (c) $a < a_c$.

Condition 1. The partition line must pass through one and only one representative of each bi-infinite sequence and may not cross the attractor elsewhere.

Indeed, as was stated in [12], the partition line for the Hénon map (with different parameters) constructed in [13] does not lead to a GP, as it passes twice through one family of HTs. However, this is not a sufficient condition, as the next four partitions (IV–VII) show. They all satisfy rule (1), but none of them is generating.

IV is given by [A_{-1}, B_{-1}, D_0, C_0] and yields $h_{29} = 0.405\,33$. For a consistency check we also looked for different periodic orbits that are encoded in the same way and found the period-15 orbits at $(0.450\,45, -0.682\,081)$ and $(-1.040\,971, 1.506\,061)$ which are both encoded by 111010101111010 (the periodic orbits were calculated using the method from [13]). V and VI are given by [A_1, B_1, D_0, C_{-1}] and [A_0, B_0, C_0, D_{-1}] and yield $h_{29}^{\mathrm{V}} = 0.3640$ and $h_{29}^{\mathrm{VI}} = 0.3624$, respectively. Partition VII is [A_0, B_0, C_{-1}, D_{-2}] and gives $h_{29} = 0.4136$. We found period-8 orbits at $(1.706\,137, -0.327\,113)$ and $(-1.762\,214, 1.778\,003)$ which both have the coding 00011111. This goes along with the lack of infinite refinement of the partition elements, when the union of all images and preimages of the partition lines is considered.

These four example partitions fail to fulfill another rule for a partition to be generating. In order to be so, the *n*th images and preimages of the partition line obviously have to

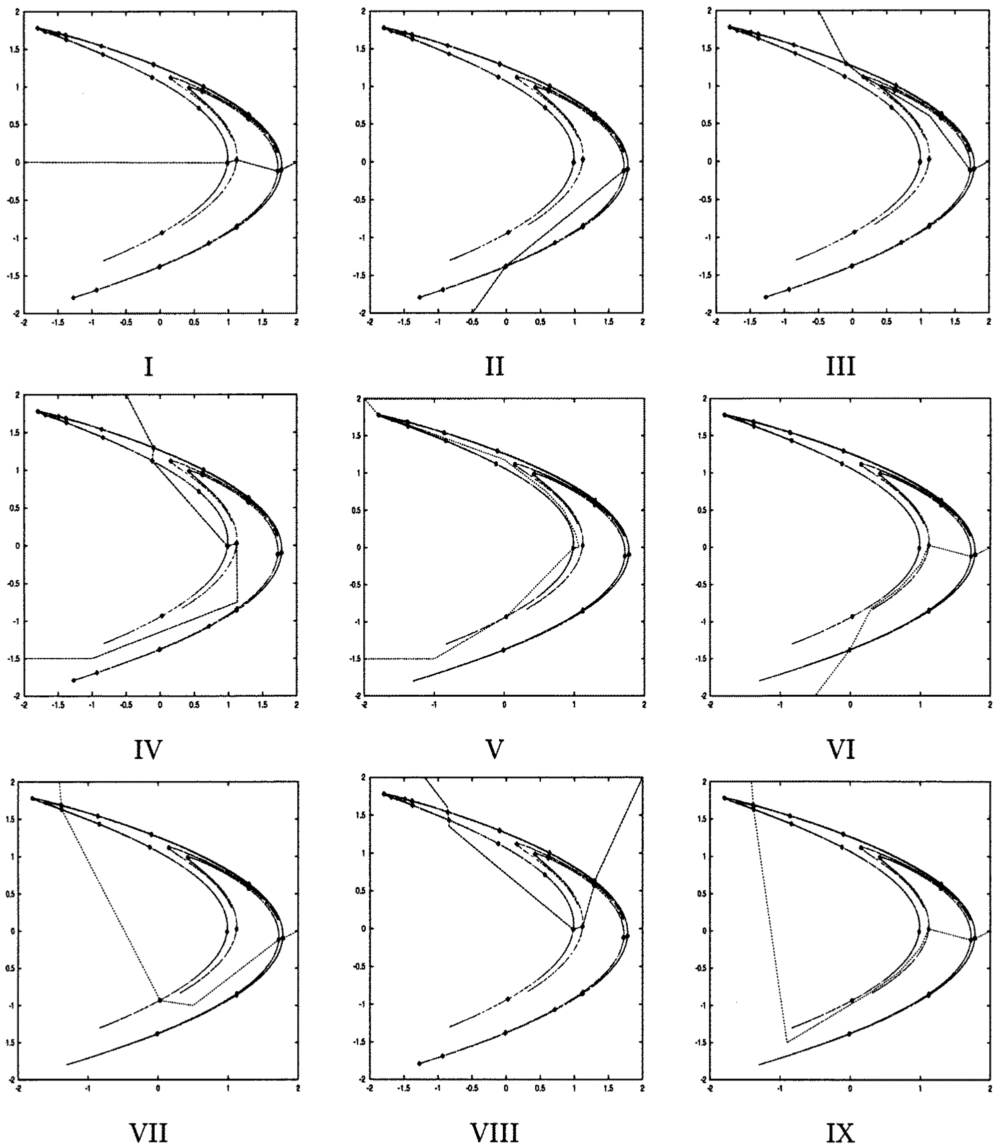


Figure 4. Different partition lines for the Hénon attractor with standard parameters $(a, b) = (1.4, 0.3)$. Partition I, VIII and IX are generating, the others are not.

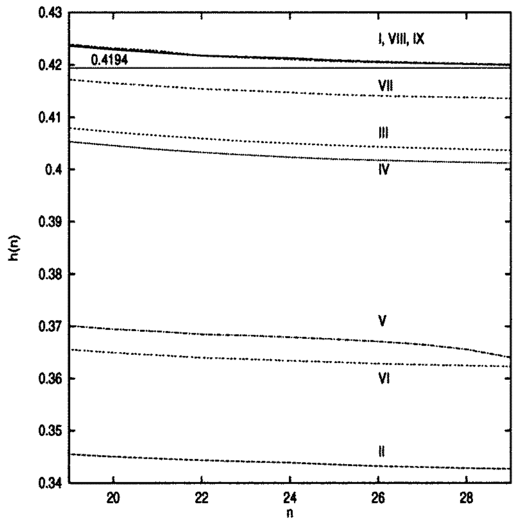


Figure 5. Incremental entropies h_n^i for the different partitions i = I–IX in figure 4. n denotes the length of the considered symbol sequence. The horizontal line indicates the metric entropy, estimated by the Lyapunov exponent. The calculation of h_n was performed with a series of length 2^{32}.

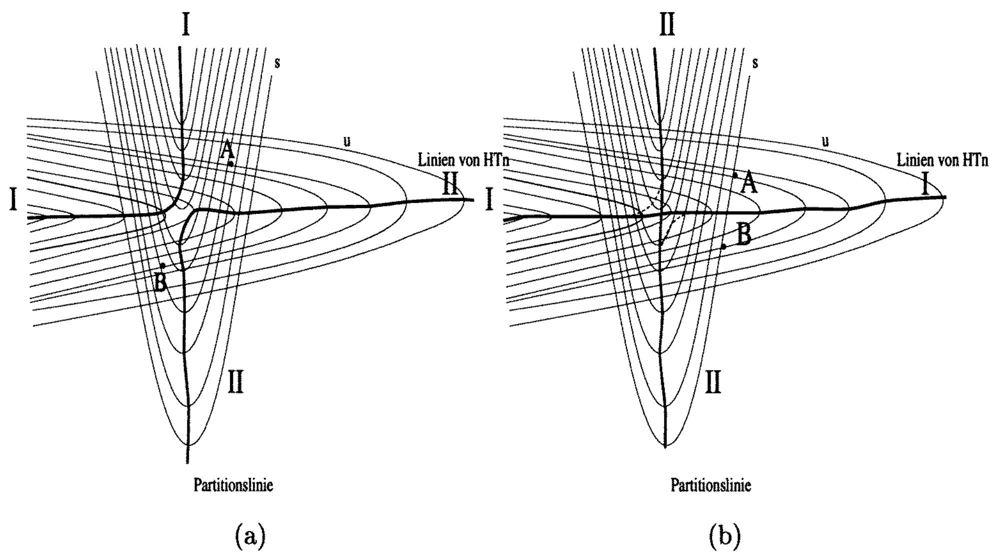
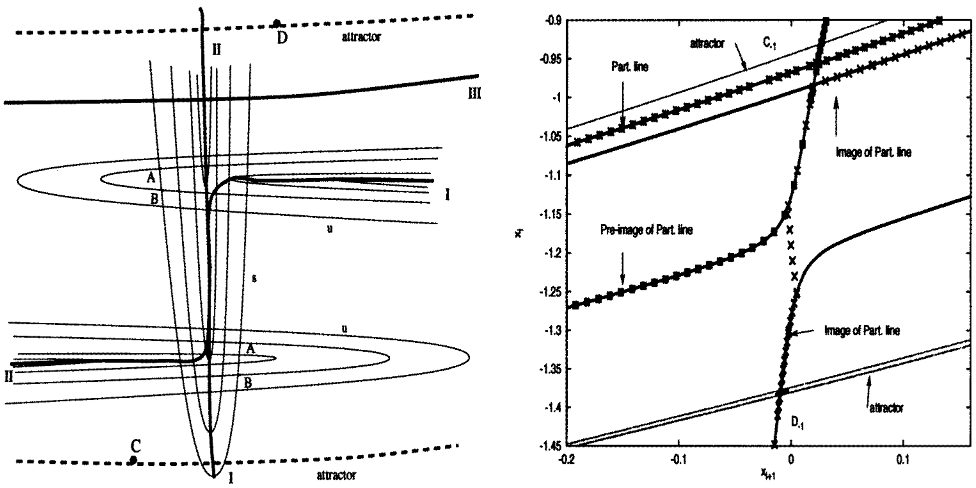


Figure 6. (a) Partition line (I) and one of its (pre)images (II) passing through an AC of HTs (crossings of parabolae of stable and unstable manifolds) in a way that does not lead to a generating partition: points A and B on one stable and unstable manifold do not become separated. (b) Partition line passing through an AC of HTs in the only way that guarantees that the partition is generating: all points A and B on one stable and unstable manifold become separated.

divide the phase space into smaller and smaller partition elements with growing n in order to separate different points on the attractor. The (pre)image of a partition line, therefore, has to intersect with the partition line itself. Following the lines of HTs in the entire phase



(a) (b)

Figure 7. (a) Schematic illustration of a situation where a failure to fulfill condition 2 can still lead to a generating partition for a dissipative system: the points C and D on the attractor are not separated through the partition line I and one of its (pre)images II, as they do not cross corresponding to condition 2. Locally the two pairs on the foliations (denoted A and B in both cases) never become separated, but for C and D on the attractor there exists another (pre)image III that separates them. (b) The occurrence of the situation schematically described in (a) for the Hénon system and the partition IX between C_{-1} and D_{-1}: the preimage partition line (squares) and the image partition line (crosses) of IX (playing the role of II and I in (a), respectively) are given. The partition line (playing the role of line III in (a)) is given by stars.

space, this can only happen at ACs! In the neighbourhood of these crossings of partition lines at ACs it has to be guaranteed that nearby points on the attractor that are on the same stable manifold are still being separated by the partition lines. This is not automatically guaranteed at these ACs! If the partition lines follow the lines of HTs and thus avoid crossing each other (indicated in figure 6(a)), it is obvious that certain points (indicated A and B in figure 6) are not separated by the partition line and its (pre)images (the situation, where the lines both jump to the other branch and turn, equally leaves pairs of points on the same stable and unstable manifold unseparated). In figure 6(b) we illustrate the situation where a separation of all the points on the same stable and unstable manifold in the neighbourhood of the AC is guaranteed. More precisely, we formulate the following rule as a *sufficient* criterium (together with condition 1) for a partition line to be generating:

Condition 2. The partition lines must pass the ACs by 'jumping' straight over to the other branch. In other words, at an AC the partition line must never turn by following or jumping over the AC.

The way the partitions IV–VII fail to satisfy condition 2 can be seen by following the partition lines along the HTs in the phase space with the help of figure 1. Partition IV jumps incorrectly over an ACs between B_{-1} and D_0 as well as after C_0. V and VI follow the lines of HTs without jumping several times, and partition VII does not satisfy condition 2 between B_0 and C_{-1} and C_{-1} and D_{-2}.

Partition VIII ([B_{-3}, D_0, C_0, A_{-2}) does satisfy the condition 2 (and 1) and is therefore generating. This can be seen from values of h_n (figure 5), where it was numerically found that the values of h_{29} correspond perfectly to the correct value h_{metric}. We also checked for different periodic orbits with the same coding up to period 25. Note that this partition is non-trivial in a sense that it is not an image or preimage of the standard partition I. Evidently all images and preimages of GP are again generating. We briefly note two things. First, condition 2 is automatically fulfilled for all AC that are not resolved in figure 1. Second, condition 2, if a necessary requirement, would determine a generating partition by its starting point. On the other hand, many starting points for a partition line when following condition 2 consequently do not lead to a generating partition as condition 1 is not fulfilled.

If we were dealing with a conservative system, condition 2 would at the same time be a necessary condition for a partition to be generating. In this case the separation of points on the same stable and unstable manifold has to be guaranteed on the entire unstable (and stable) foliation, as the attractor fills the whole phase space (this has been investigated in more detail in [5]). In the case of dissipative systems, however, there are situations where condition 2 is not satisfied and the partition is still generating. This is the case in partition IX ([A_0, B_0, C_0, D_{-2}]): condition 2 is not satisfied between C_0 and D_{-2}, the partition lines follows the lines of HT and do not jump (compare with figure 1). Nevertheless, the entropy h_n converges to the correct value (figure 5), and we did not find any two different periodic orbits up to length 25 that are encoded in the same way. Pairs of points on the same stable and unstable manifold that are not separated at an AC in this case are simply not on the attractor. Any pair of points on the attractor becomes separated by another (pre)image of the partition. In the case of partition IX, the situation is schematically illustrated in figure 7(a). Furthermore, in figure 7(b) we illustrate the occurrence of this situation for partition IX. We show part of the original partition line (taking the role of line III in (a)) and part of its image and preimage, that takes the role of partition line II and I in (a), respectively. It is obvious that this situation also occurs at image locations.

In these cases it has to be checked whether points on the attractor close to such an AC are separated or not.

We conclude that generating partitions are not uniquely defined. Applying just the criterion that partition lines intersect the attractor at homoclinic tangencies leaves us with an infinite set of possibilities, among which most lines do not lead to a generating partition. The relevant additional criterion comes from the way the generating partition lines pass through the avoided crossings outside the attractor, which becomes applicable *only* if one knows the structure of lines of homoclinic tangencies in the full phase space. A possible way to obtain this essential information has been introduced in [8].

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